Mathematical model of process of the protein synthesis

Valerii Dryuma*

Institute of Mathematics and Informatics, AS RM

valery@dryuma.com; valdryum@gmail.com

Abstract. *New approach to study properties of the system of differential equations of the first order which describe process of the protein synthesis is proposed. Accurate results concerning properties of its solutions depending on correlations between the parameters of the model are obtained.*

1 Introduction

A mathematical model, which describes the process of protein synthesis has been considered in [1]. In accordance with this model, the synthesis of protein in the cell depends on the ratio between the concentration of ribonucleic acid $x(t)$, the enzyme $y(t)$ and of operator $z(t)$ in a cell at the moment t . Its dynamics in time is described by the solutions of the system of differential equations

$$\frac{d}{dt}x(t) = \frac{c}{a + bz(t)} - kx(t),$$
$$\frac{d}{dt}y(t) = ex(t) - fy(t),$$
$$\frac{d}{dt}z(t) = \nu\, y(t) - \alpha\, x(t) \tag{1}$$

with the parameters $a,\ b,\ c,\ k,\ e,\ f,\ \nu,\ \alpha.$

This system is equivalent to the ODE of second order of type

$$\frac{d^2}{da^2}b(a) + A_1\left(\frac{d}{da}b(a)\right)^3 + 3\,A_2\left(\frac{d}{da}b(a)\right)^2 + 3\,A_3\,\frac{d}{da}b(a) + A_4 = 0, \tag{2}$$

where the coefficients A_i depends on the parameters and have a form

$$A_1 = 0,\ 3\,A_2 = \frac{-ka + b(a)}{b(a)\,(ka + b(a))},\ 3\,A_3 = \frac{\left(-kf - k^2\right)a + (-f + 3\,k)\,b(a)}{b(a)\,(ka + b(a))},$$

$$A_4\,(b(a))^2\,c\,(ka + b(a)) =$$
$$= \left(k^3 fb\alpha - k^3 b\nu\,e\right)a^4 + \left(3\,k^2 fb\alpha - 3\,k^2 b\nu\,e + k^3 b\alpha\right)b(a)a^3 +$$
$$+ \left(3\,k^2 b\alpha - 3\,kb\nu\,e + 3\,kfb\alpha\right)(b(a))^2\,a^2 + \left((3\,kb\alpha - b\nu\,e + fb\alpha)\,(b(a))^3 - k^2 fcb(a)\right)a +$$
$$+ (b(a))^4\,b\alpha + \left(-fkc + 2\,k^2 c\right)(b(a))^2\,.$$

Obtaining of exact solutions of system (1) and the corresponding ODE of second order (2) is a difficult problem, but some particular solutions can be obtained by using the theory of invariants.

Invariants of Liouville-Tress-Cartan equations for (2) are being built with the help of coefficients A_i. The correlation between them depend on the parameters and can be used to study properties of solutions of the equation.

From another point of view equation of the type (2) can be studied by the methods of Riemannian geometry.

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With this purpose the equation (2) is represented in the form of the system of equations

$$\ddot{b} + A_4\dot{a}^2 + 2A_3\dot{a}\dot{b} + A_2\dot{b}^2 = 0,\ \ddot{a} - A_3\dot{a}^2 - 2A_2\dot{a}\dot{b} - A_1\dot{b}^2 = 0,$$

which has the form of the geodesic lines of the two-dimensional affine-connected space in the local coordinates a, b. Its properties can be studied on the basis of a four-dimensional Riemannian space in the local coordinates $(a, b, z, \tau$ and equipped with the metric

$$ds^2 = 2(zA_3 - \tau A_4)(da)^2 + 4(zA_2 - \tau A_3)dadb + 2(zA_1 - \tau A_2)(db)^2 + 2dadz + 2bd\tau.$$

Here we will consider the properties of the model (1) and its simplified variant by the algebraic - geometrical methods.

2 Implicit equation

Some properties of synthesis of protein in cell can be studied on base of simplified system of equations

$$\frac{d}{dt}x(t) = \frac{c}{\alpha + \mu\, y(t)} - kx(t),$$

$$\frac{d}{dt}y(t) = ex(t) - \beta\, y(t), \tag{3}$$

which follows from the system (1) at the condition $\dot{z} = 0$.

Analytic representation of solutions of the system (3) is still a difficult problem and for the study its properties can be used theory of implicit equations.

The system of equations has parametric solution of the form

$$b(a) = \frac{d}{dt}y(t),\ a = y(t),$$

or

$$t = \int (b(a))^{-1}\, da + C1,\ y(t) = a,$$

where the function $b(a)$ satisfies the equation

$$\frac{d}{da}b(a) = \frac{((-\mu\,\beta - k\mu)\,a - \alpha\,\beta - k\alpha)\,b(a) - k\mu\,a^2\beta - k\alpha\,\beta\,a + ce}{b(a)\,(\alpha + \mu\,a)} \tag{4}$$

and

$$x(t) = \frac{\frac{d}{dt}y(t) + \beta\,y(t)}{e}.$$

Theorem 1 *A homogeneous system of equations which corresponds to the equation* (4) *has the form*

$$\frac{d}{dt}x(t) = (4\,\alpha\,y + \alpha\,(\beta + k)\,x)\,z + 3\,\mu\,xy + \mu\,(\beta + k)\,x^2,$$

$$\frac{d}{dt}y(t) = 4\,cez^2 + (-3\,\alpha\,(\beta + k)\,y - 4\,k\alpha\,\beta\,x)\,z - \mu\,y^2 - 3\,\mu\,(\beta + k)\,xy - 4\,k\mu\,\beta\,x^2,$$

$$\frac{d}{dt}z(t) = z^2\alpha\,(\beta + k) + (\mu\,(\beta + k)\,x - \mu\,y)\,z.$$

Corresponding p.d.e. to the function $z = z(x, y)$

$$\left(\frac{\partial}{\partial x}z(x,y)\right)\left(4\,\alpha\,yz(x,y) + 3\,\mu\,xy + \alpha\,(\beta + k)\,z(x,y)x + \mu\,(\beta + k)\,x^2\right) +$$

$$+\left(\frac{\partial}{\partial y}z(x,y)\right)\left(-\mu\,y^2 + 4\,cez^2 - 4\,xk\alpha\,\beta\,z(x,y) - 3\,\alpha\,(\beta + k)\,yz - 4\,k\mu\,\beta\,x^2 - 3\,\mu\,(\beta + k)\,xy\right) -$$

$$-z(x,y)\mu\,(\beta + k)\,x - (z(x,y))^2\,\alpha\,(\beta + k) + \mu\,yz(x,y) = 0$$

after the (u, v)-transformation

$$z(x, y) = u(x, t),\ y = v(x, t),\ z_x = u_x - \frac{u_t}{v_t} v_x,\ z_y = \frac{u_t}{v_t},$$

$$v(x, t) = t\omega_t - \omega,\ u(x, t) = \omega_t,\ \omega = \omega(x, t)$$

has solution of the form $\omega(x, t) = A(t)x$ with the function $A(t)$ satisfying an implicit first order ODE

$$(-\alpha\, A(t)t - ce + \alpha\beta t + \alpha\, kt)\left(\frac{d}{dt}A(t)\right)^2 +$$

$$+\Big(\mu\, kt + k\alpha\,\beta - \alpha\, kA(t) + \mu\,\beta t - \alpha\,\beta\, A(t) + \alpha\,\left(A(t)\right)^2 - \mu\, A(t)t\Big)\,\frac{d}{dt}A(t) -$$

$$-\mu\,\beta\, A(t) + \mu\,\left(A(t)\right)^2 - \mu\, kA(t) + \beta\,\mu\, k = 0. \tag{5}$$

Algebraic curve, $F(x, y, p) = 0$, which corresponds to the equation (5) after the change of variables $t = x,\ A(t) = y,\ \dot{B} = p$

$$(-\alpha\, x, yce + \alpha\,\beta, x + \alpha\, xk)\,(p)^2 +$$

$$+\Big(\mu\, xk + k\alpha\,\beta - \alpha \rightarrow .y) + \mu\,\beta x - \alpha\,\beta\, y) + \alpha\,(z)^2 - \mu\, x, y\Big).$$

$$-\mu\,\beta\, y) + \mu\,(z)^2 - \mu\, ky + \beta\,\mu\, k = 0 \tag{6}$$

is a curve of genus $g = 1$ relative variables (p, y).

Properties of solutions of the equation (5) depend from singular points of the curve (6) and studies of their characteristics may be useful for the theory of protein synthesis.

Theorem 2 *Solutions of the system of equations*

$$\dot{x} = F_p,\ \dot{y} = pF_p,\ \dot{p} = F_x + pF_y$$

connected with the equation (5) are expressed through solutions of the first order differential equation $\frac{dy}{dx} = \frac{Q_5(x,y)}{P_2(x,y)}$ where Q_5 and P_2 are polynomials of degree 5 and 2 . Their properties can be investigated by the method of (u, v)-transformation which developed by author [2].

3 3-dim model

Theorem 3 *A full system of equations (1) can be investigated on base of the system of equations*

$$\frac{d}{dt}\xi\,(t) = (-5/6\, k\xi + 1/6\,\xi f - 1/6\,\xi\alpha + c\theta)\,\rho^2 + 1/3\,\xi\alpha\, a\theta\rho + 1/3bg\,\xi\theta\eta,$$

$$\frac{d}{dt}\eta\,(t) = (1/6\,\eta k - 5/6\, f\eta - 1/6\,\eta\alpha + e\xi)\,\rho^2 + 1/3\,\eta\alpha\, a\theta\rho + 1/3\, bg\theta\eta^2,$$

$$\frac{d}{dt}\theta\,(t) = -1/6\,\theta\,(-k - f - 5\,\alpha)\,\rho^2 - 2/3\,\theta^2\alpha\, a\rho - 2/3\,\theta^2 g\eta b,$$

$$\frac{d}{dt}\rho\,(t) = (1/6\, k + 1/6\, f - 1/6\,\alpha)\,\rho^3 + 1/3\,\theta\rho^2\alpha\, a + 1/3\, gb\theta\rho\eta, \tag{7}$$

which is homogeneous extension of the system (1) written in equivalent form of cubical system of equations. The correlation between the variables of both systems of equations related by projective transformations

$$\xi(t) = x(t)\rho(t),\ \eta(t) = y(t)\rho(t), \theta(t) = z(t)\rho(t).$$

Properties of solutions of the system of equations (7) can be investigated by the method of exponents of Kovalevskaya [3] or by the method [2]. As example we bring solutions of the form

$$\left\{\eta\,(t) = \frac{B}{\sqrt{t}}, \theta\,(t) = \frac{C}{\sqrt{t}}, \rho\,(t) = \frac{E}{\sqrt{t}}, \xi\,(t) = \frac{A}{\sqrt{t}}\right\},$$

where

$$C = RootOf\left((2\,bgec + 4\,fcegb + 4\,\alpha\,cegb)\,\_Z^2 + \left(fE\alpha\,a + 2\,fE\alpha^2 a + 2\,f^2E\alpha\,a\right)\_Z + 6\,f\alpha\right),$$

$$E = RootOf\left(6 + (1 + 2\,f + 2\,\alpha)\,\_Z^2\right),$$

$$B = 2\,\frac{ecRootOf\left((2\,bgec + 4\,fcegb + 4\,\alpha\,cegb)\,\_Z^2 + \left(fE\alpha\,a + 2\,fE\alpha^2 a + 2\,f^2E\alpha\,a\right)\_Z + 6\,f\alpha\right)}{f},$$

$$A = 2\,cRootOf\left((2\,bgec + 4\,fcegb + 4\,\alpha\,cegb)\,\_Z^2 + \left(fE\alpha\,a + 2\,fE\alpha^2 a + 2\,f^2E\alpha\,a\right)\_Z + 6\,f\alpha\right),$$

containing useful information about properties of considered model of protein synthesis.

References

[1] S. V. Fomin, M.B. Berkenblit: Mathematical Problems in Biology. *Izd. "Nauka", Glavnaya redactiya fiz.mat. lit.*, Moskow, 1973.

[2] V. Dryuma: On the theory of the first order systems of differential eqiations. *International Conference on Differential Equations and Dynamical Systems, Abstracts*, Russia, Suzdal, July 2-7, p.205-206, 2010.

[3] Alan Goriely: Integrability and Nonintegrability of Dynamical Systems. *World Scientific New Jersei, London, Singapore, HongKong*, 2001.