Parallel computations in insurance business optimization

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Abstract. The paper considers parallel computing opportunities for optimization of insurance bussiness. Evolution of the capital of the company is modelled by a risk process or autoregression process with dividend subtraction. We consider positional dividend strategies as functions of a current capital. Strategies are evaluated by two criteria: average collected dividends and a ruin probability on a finite time interval. The values of the criteria are evaluated by Monte-Carlo method. To get more exact values of the ruin probability a successive approximation method is applied for solution of integral equations satisfied by the ruin probability. We give a detailed numerical comparison of two dividend strategies: a percentage of the current capital and maximization of the current dividends subject to a bound on the ultimate ruin probability. Parallel versions of the Monte Carlo simulation method and the successive approximation methods implemented on a cluster of several personal computers having two or four cores (up to twenty cores in total) are compared.

Keywords
Actuarial mathematics, risk process, autoregression process, ruin probability, optimization of dividends, successive approximation method, Monte Carlo method, parallel computing.

Introduction

Work of insurance companies and other financial institutions should be evaluated in terms of profitability and risk. This article considers average collected dividends as measure of profitability and ruin probability on a fixed time interval as measure of risk. Evolution of a company‘s capital is modelled by a risk process or an autoregression process with dividend subtraction. To visualize results of decision, simulation results are plotted in the "income - risk" plane.

Estimating ruin probability of an insurance company is an important practical problem, but it appears to be difficult theoretical and computational one. Although ruin probability can, in general, be estimated by statistical tests (Monte Carlo method, i.e. modeling stochastic trajectories of the company's reserves evolution), but it requires an astronomical number of simulations for estimating small probabilities [2]. Therefore, actuarial mathematics pays great attention to analytical and numerical methods for assessing the ruin probability [1 - 4]. It is known that ruin probability satisfies certain integral equations. One of common approaches is the method of successive approximations (MSA) [5 - 10]. At each step, MSA calculates one or two-dimensional integrals over a large range, that is a time-consuming computational procedure. At each iteration integration can be carried out either by quadrature formulas or by means of Monte Carlo method. It is therefore natural to implement MC and MSA methods on HPC parallel computing systems. MC method has natural parallelism: simulations can be performed independently on different processors. MSA is a functional iteration method thus also admits parallelization.

1 The problem

Risk process as a model of an insurance company. Stochastic model of the company‘s capital evolution reads as follows [1 - 9]:

$$x_t = u + \int_0^t (c - D(x_s))ds - S_t, \; t \geq 0,$$

where $x_0 = x \geq 0$ is initial capital; $S_t = \sum_{k=1}^{N_t} z_k$ is aggregate random insurance claim; z_k is a ramdom claim at time moment t_k with a cumulutive distribution function (CDF) $\overline{F}_{t_k}(\cdot)$; N_t is the number of random claims before time moment t; c is aggregate insurance premium per unit of time; $D(\cdot)$ is a function representing intensity of dividend payments depending on current reserves, $0 \le D(\cdot) \le c$. Here $D(\cdot)$ can be understood as the dividend control function. For exsample,

$$D(U) = \begin{cases} 0, & U \ge B(U), \\ c, & U < B(U), \end{cases} \quad (1)$$

where $B(\cdot)$ is some monotonically increasing function, called a dividend barrier [1].

Modeling evolution of capital by means of an autoregressive process [11]. As mentioned, the autoregression modell with dividend subtraction can be used as a model of companye's capital dynamics. It is assumed that revenues and expenses have two components, deterministic and stochastic. The deterministic part is a function of current capital and stochastic one is a stationary random process. For example, the deterministic component can simulate estimated profit (income minus expenses) of the organization and the random part is stochastic deviations from the plan.

Let the equation of capital variation (in discrete time) be

$$x_{t+1} = f(x_t, \xi_t) - D(x_t), \qquad t = 0,1,\dots, \quad (2)$$

where $x_0 = x \ge 0$ specifies the initial state, $D(\cdot)$ denotes planned dividends as a function of a current capital x_t, $\{\xi_t\}$ denotes random factors affecting evolution of the capital.

Dividend policy management problem. Control strategy $D(\cdot)$ is estimated on the basis of two indicators, profitability (total dividends) and risk (probability) of ruin. We consider three interrelated tasks:

predicting risk of insolvency of a financial organization under management strategy $D(\cdot)$;
company's functioning parametric optimization subject restrictions on ruin probability (discussed in [12]);
optimization of dividends positional control under a constraint on ruin probability.

Positional control of dividends. Let $\varphi_d(x,t)$ denotes ruin probability on the time interval $[0,t]$ with constant control $D(\cdot) \equiv d$. It is obvious, that functions $\varphi_d(x,t)$ are nondecreasing in x and do not increase in d, t, and $\varphi_d(+\infty,t) = 1$, $\varphi_d(x,+\infty) = 0$. Let ε denotes an upper bound for ruin probability, then $\delta = 1-\varepsilon$ denotes a lower bound for nonruin probability. Thus for any ε, $0 < \varepsilon < 1$, and $d \ge 0$ there exists minimal solution $x_t(\varepsilon,d)$ of the inequality, $\varphi_d(x,t) \ge 1-\varepsilon$; for all $\varepsilon \ge 1-\varphi_0(x,t)$ there exists a maximal solution $d_t(\varepsilon,x)$ of the inequality $\varphi_d(x,t) \ge 1-\varepsilon$. Cumulative dividends $D_t(x,d)$ are stochastic value, because deterministic dividends d could be collected not over all period of time t, but only before ruin, that takes place at time $\tau \le t$. It is obvious that $D_t(x,d) \le dt$. However, if the nonruin probability over time t is greater or equal than $(1-\varepsilon)$, then the average quantity $ED_t(x,d)$ over the realizations $D_t(x,d)$ is not smaller than $(1-\varepsilon)dt$, thus $(1-\varepsilon)dt \le ED_t(x,d) \le dt$. Symbol E in the expression $ED_t(x,d)$ denotes mathematical expectation over all possible trajectories of the process. If ε is close to zero, then by maximizing dividends d under a constraint $\varphi_d(x,t) \ge 1-\varepsilon$ we indirectly (approximately) maximize the average value $ED_t(x,d)$ of collected dividends. By virtue of monotonicity of $\varphi_d(x,t)$ with $\varepsilon \ge 1-\varphi_0(x,t)$, we get $d_t(\varepsilon,x) = \arg\max\{d \ge 0 : \varphi_d(x,t) \ge 1-\varepsilon\}$. Obviously, the dividend strategy

$$d^1_\tau(\varepsilon,x) = \begin{cases} 0, & \varphi_0(x,t) < 1-\varepsilon, \\ d_t(\varepsilon,x), & \varphi_0(x,t) \ge 1-\varepsilon, \end{cases} \qquad 0 \le \tau \le t,$$

is almost optimal *stationary* company's strategy subject to a constraint on ruin probability (ε) and with an initial capital x. This strategy does not take dividends in case when the initial capital x is small $\varphi_0(x,t) < 1-\varepsilon$. And when $\varphi_0(x,t) \ge 1-\varepsilon$, the strategy takes maximal possible constant dividends.

Since current capital of the company is permanently changing, it is natural to use a variable (quasistationary) dividend policy, viz. at the time $\tau < t$ we can use control

$$d^2_\tau(\varepsilon,x_\tau) = \begin{cases} 0, & \varphi_0(x_\tau,t') < 1-\varepsilon, \\ d_{t'}(\varepsilon,x_\tau), & \varphi_0(x_\tau,t') \ge 1-\varepsilon, \end{cases} \qquad 0 \le \tau \le t. \quad (3)$$

where t' is a fixed time parameter and $0 < t' \le t$. If current assets x_τ are not enough to guarantee nonruin probability $(1-\varepsilon)$ over the next t' periods of withdrawing constant dividends, then no dividends are taken. If current assets are sufficient, i.e. $\varphi_0(x_\tau, t') \ge 1-\varepsilon$, then we withdraw such amount of dividends $d_{t'}(\varepsilon, x_\tau)$ that if they were taken over additional time t', then non-ruin probability on the time interval $[\tau, \tau+t']$ would be greater than or equal to $(1-\varepsilon)$.

With this strategy there is a chance to collect more dividends than with the stationary one, still keeping ruin probability below ε (taking t' of the same order as t).

Let's also consider a simple (interest) strategy

$$d_\tau^3(x_\tau) = \alpha x_\tau,\ 0 < \alpha \ll 1. \qquad (4)$$

Integral equations for ruin probability. Estimating company's ruin probability is a complex theoretical and computational problem. Although ruin probability, in principle, can be estimated by statistical tests like MC (simulation of trajectories of stochastic evolution of the company's reserves), but to estimate small probabilities this method requires a very large number of tests.

Nonruin probability $\varphi(u,t)$ of an insurance company in time interval $[0,t]$ with initial reserve x satisfies the integral equation [10]:

$$\varphi(x,t) = \int_0^t \int_0^{U(x,\tau)} \varphi\left(U(x,\tau)-z, t-\tau\right) dF(z,\tau) + \left(1-F(\infty,t)\right), \qquad (5)$$

with boundary conditions $\varphi(x,0)=1$, $\varphi(+\infty,t)=1$, where $F(z,\tau)$ is a joint distribution function of arrival times τ and claim sizes z for mutually independent insurance claims, $F(z,0)=0$; function $U(x,t)$ is a solution of the Cauchy problem:

$$\frac{dU}{dt} = c - D(U), \qquad U(0)=x, \qquad t \ge 0.$$

2 The method

Dividend strategies are compared according to criteria of profitability and risk, namely, by average collected dividends and ruin probability on a finite time interval t, that were estimated by statistical simulation (Monte Carlo).

For estimation of ruin probability a method of successive approximations is proposed. At each step of the method one has to calculate one or multi-dimensional integrals over a large domain. It is a time consuming computational procedure. Integration at each iteration can be carried out by quadrature formulae or by means of Monte Carlo method. It is therefore natural to implement MC and MSA methods on HPC parallel computing systems. MC method has a natural parallelism: simulations can be performed independently on different processors. MSA is a functional iteration method, so it also admits parallelization.

Parallel method of successive approximations [10]. In general form, equation (5) (and other similar equations) can be written as

$$\varphi(x,t) = A_{x,t}\varphi(\cdot,\cdot), \qquad (6)$$

where $A_{u,t}$ is a corresponding integral operator. It is easy to see that $A_{u,t}$ in (5) is contracting in the following sense :

$$\left|\varphi_1(x,t)-\varphi_2(x,t)\right| \le \left(1-F(\infty,T)\right) \sup_{\substack{0\le x<\infty,\\ 0\le t'\le T}} \left|\varphi_1(x,t')-\varphi_2(x,t')\right|$$

for any bounded monotone (increasing in x and decreasing in t) functions $\varphi_1(x,t), \varphi_2(x,t)$, $x\in[0,\infty)$, $t\in[0,T]$.

The method of successive approximations for the solution of equation (5), (6) has the form:

$$\varphi^{k+1}(x,t) = A_{x,t}\varphi^k(\cdot,\cdot), \qquad 0 \le \varphi^0(x,t) \le 1, \qquad k=0,1,\ldots,$$

where $\varphi^0(x,t)$ is some initial function, nondecreacing in x and decreacing in t. This method was studied in details in [5 – 10]. If $\varphi^0(x,t)\equiv 1$, the sequence of approximations $\left\{\varphi^k(u,t)\right\}$ decreases monotonically and converges to the solution from above, and if $\varphi^0(x,t)\equiv 0$, then $\left\{\varphi^k(x,t)\right\}$ increases monotonically and converges to the solution from below. Norm of the difference between approximations from above and below gives us an estimate for accuracy of an approximate solution. In practice, calculations are performed within a certain region $\{0 \le x \le x_\infty, 0 \le t \le t_{\max}\}$ such, that $\varphi(x,t) = 1$ with $x \ge x_\infty$. If functions $\varphi^k(\cdot,\cdot)$ and $\varphi^{k+1}(\cdot,\cdot)$ are determined by thier values at nodes (x_i, t_j) of a two-dimensional grid, and values at the remaining points are found by interpolation, then the value $\varphi^{k+1}(x_i, t_j)$ at the node (x_i, t_j) could be found by knowing $\varphi^k(\cdot,\cdot)$ regardless of the calculations at other nodes, simply by calculating the two-dimensional integrals

$$\varphi^{k+1}(x_i, t_j) = \int_0^{t_j} \int_0^{U(x_i,\tau)} \varphi^k\left(U(x_i,\tau) - z, t_j - \tau\right) dF(z,\tau) + \left(1 - F\left(\infty, t_j\right)\right). \quad (7)$$

Thus, calculations in MSA can be parallelized over (x_i, t_j), i.e. at each iteration k for different pairs (x_i, t_j) computations can be carried out independently and in parallel, and at the end of iteration, all values $\varphi^{k+1}(x_i, t_j)$ are collected into one array, which is then transmitted to all computational cores to perform a new iteration. Calculations are started from $\varphi^0 \equiv 1$ and from $\varphi^0 \equiv 0$ for accuracy control.

Parallel Monte Carlo method. We simulate N trajectories of stochastic evolution of an insurance company reserves on a given time interval $[0, t_{\max}]$ for each value of initial capital $x \le x_\infty$ and calculate a share $p_N(x,t)$ of survived up to time $t \le t_{\max}$ trajectories and amount of collected dividends [11]. For each x calculations are uniformly distributed between computational cores. There is no data exchange during iteration, but at the end of it the array of trajectories is passed to one core and the function $p_N(x,t)$ is constructed. The accuracy of the Monte Carlo method can be estimated by means of Hoeffding's inequality

$$\Pr\left\{\left|p_N(x,t) - \varphi(x,t)\right| \ge \delta\right\} \le 2e^{-N\delta^2/2}$$

so (10^{-k})-confidence bound for $\left|p_N(x,t) - \varphi(x,t)\right|$ is $\delta_k(N) = \sqrt{2\left(k\ln 10 + \ln 2\right)} \big/ \sqrt{N}$. Values of 0.01-confidence bounds $\delta_2(N)$ are given in Table 3.

3 Numerical experiments

Numerical experiments on estimating ruin probability. Calculations were performed on a mini-cluster DISOPT of Institute of Cybernetics, equipped with Intel Core2Quad Q9550 processors, for an insurance model with the following parameters:

$$U(x,t) = x + ct,\ F(z,\tau) = \left(1 - e^{-z/\mu}\right)\left(1 - e^{-\alpha\tau}\right),\ \alpha = 0.2,\ \mu = 10,\ c = 1,\ t_{\max} = 50,\ x_\infty = 200.$$

Note that $\alpha\mu/c = 2 > 1$, so the insurance company is certainly (with probability 1) bankrupt over the infinite time interval regardless of its initial state. Figure 1 demonstraits a graph and Table 1 shows values of approximation $\varphi^{31}(x,t)$. The results of solving equation (5) by method (7) on one, four and eight cores of DISOPT cluster are given in Table 2. Figure 3 shows the results of solving the same problem by Monte Carlo method on a mini-cluster NaUKMA-214, that is equipped with Intel E1400 Core2Duo processors (in total 20 cores).

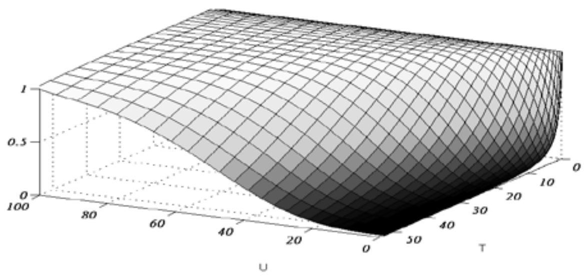

Fig. 1 $\varphi^{31}(u,t)$

Table 1. $\varphi^{31}(u,t)$

t\u	*0*	*20*	*40*	*60*	*80*	*100*	*120*	*140*	*160*	*180*	*200*
0	1	1	1	1	1	1	1	1	1	1	1
10	0,26759	0,706627	0,899259	0,968347	0,99083	0,997472	0,999335	0,999832	0,99996	0,99999	0,999997
20	0,130629	0,488484	0,74704	0,888953	0,955795	0,98361	0,994288	0,998107	0,999414	0,999816	0,999947
30	0,076697	0,3438	0,600573	0,784424	0,894728	0,952485	0,979963	0,992015	0,997025	0,998898	0,999637
40	0,048861	0,245944	0,475451	0,673914	0,816032	0,904274	0,953581	0,978807	0,990947	0,996209	0,998639
50	0,033034	0,179616	0,374796	0,568926	0,72831	0,841653	0,913895	0,955927	0,978839	0,990123	0,996095

Table 2. Efficiensy study for a parallell MSA

Number of cores	*1*	*4*	*4*	*7*	*8*
Number of iterations	31	31	42	31	31
Accuracy	3.0611×10^{-8}	3.0611×10^{-8}	4.4409×10^{-16}	3.0611×10^{-8}	3.0611×10^{-8}
Time, sec.	1704.60	668.10	864.48	407.94	386.02

Table 3. Accuracy of Monte Carlo method

Number of trials N	*N=1000*	*N=10000*	*N=100000*
Achived accuracy	0.04	0.015	0.005
Theoretical accuracy $\delta_2(N)$	0.1029	0.0326	0.0103
Time, sec.	20.19	123.55	1195.9

Optimization of a financial institution dividend strategy subject to a constrain on ruin probability. Further, strategies (3), (4) were compared by profitability and risk criteria, namely, by average collected dividends and ruin probability on a finite time interval t, which were estimated by statistical modeling (Monte Carlo) with a number of tests $N=10000$. For each strategy, cumulative dividends and ruin probability were calculated as a function of the initial capital and time (Fig. 2 and 3). The results were plotted on “dividend yield - ruin probability" graph (Fig. 4, 5). Fig. 5 shows, in particular, that strategy (3), maximizing current dividends with bounded by ε ultimate ruin probability, dominates the 10 percent strategy in a wide range of parameter ε. The value of business reliability ε can be chosen by means of diagram (Fig.3) depending on the initial company’s assets.

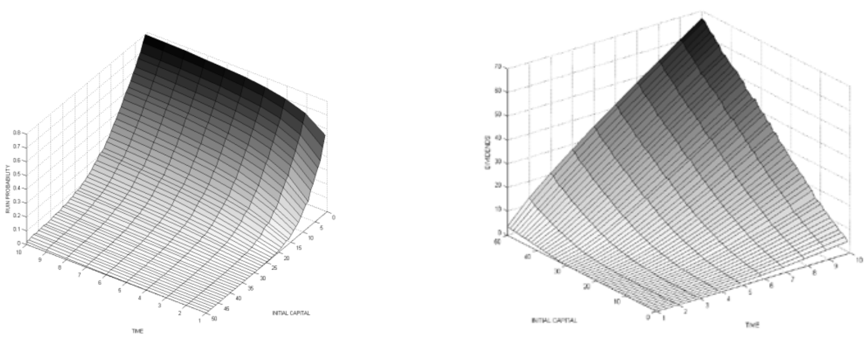

Fig. 2.
Figure 2 shows the ruin probability (left) and collected dividends (right) as functions of the initial capital and time for a nonstationary dividend strategy (3) with parameters $\varepsilon=0.05,\ t'=10$.

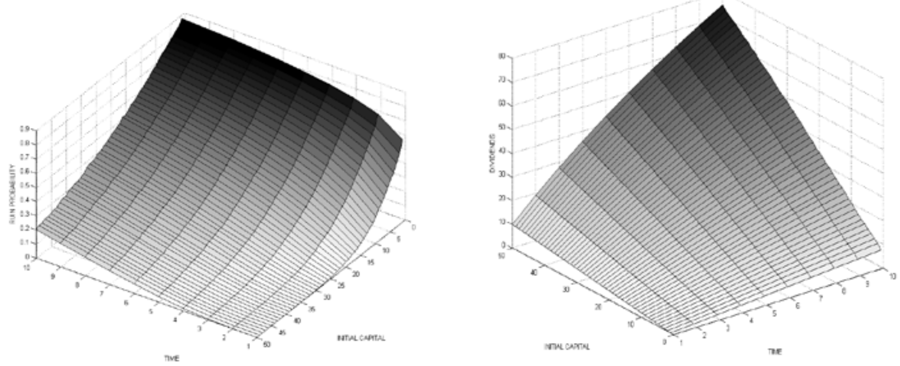

Fig. 3.

Figure 3 shows ruin probability (left) and collected dividends (right) as functions of the initial capital and time for simple interest strategy (4) with dividend rate $\alpha = 0.2$.

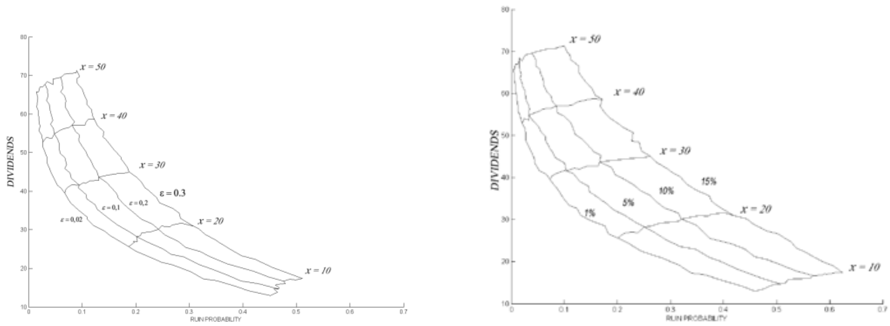

Fig. 4.

Figure 4 shows dependence of collected dividends and ruin probability for a nonstationary dividend strategy (3) and parameters $\varepsilon \in \{0.02,\ 0.1,\ 0.2,\ 0.3\}$, $100\%\alpha \in \{1\%,\ 5\%,\ 10\%,\ 15\%\}$, $t' = t = 10$.

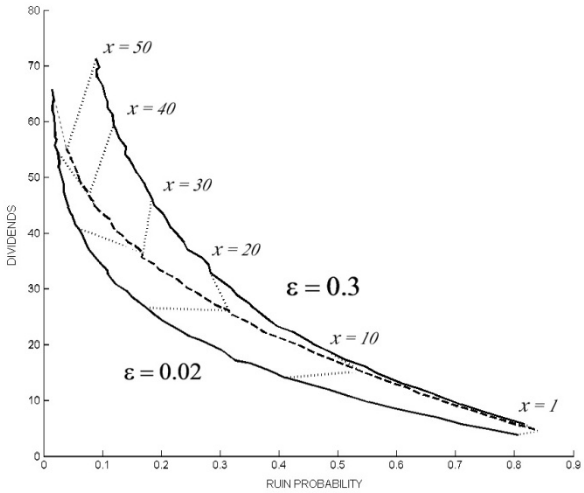

Fig. 5. Comparison of constant interest and quasisatationary dividend policies

Figure 5 shows comparison of interest ($\alpha = 0.1$, dashed line) and quasistationary dividend strategies ($\varepsilon = 0.02, 0.3$, solid lines) for different values of initial capital $x \in \{1, 10, ..., 50\}$; $t' = 10$. Point segments connect positions corresponding to the same value of initial capital.

4 Conclusions

The paper shows a possibility of parallelizing the method of successive approximations (MSA) and the method of statistical trials (MC) to estimate risk (probability) of an insurance company bankruptcy. Numerical experiments were made on mini-clusters consisting of several (up to ten) personal computers with two or four cores each. For the MSA time needed to solve the problem (finding a solution of two-dimensional integral equation for the survival probability as a function of initial capital and time interval) on eight cores is about five times less than when using single core. Time consumed by MC method is decreased approximately inversely proportionally to a number of cores. MSA allows us to solve the problem with any needed accuracy, for example, while solving problems with accuracy 10^{-7} on a cluster of two machines with processors Intel Core2Quad Q9550 (eight cores) is about 6 minutes. Such accuracy is practically unacheavable for MC, for example, when the number of trials was100000 for a cluster of ten machines with processors Intel E1400 Core2Duo (total 20 cores) only 5×10^{-3}-accuracy was achieved for time of about 20 minutes.

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